JOURNAL OF APPROXIMATION THEORY **84**, 351–366 (1996)
ARTICLE NO. 0023

A Discrete Korovkin Theorem and BKW-Operators

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Communicated by Dany Leviatan

Received August 8, 1994; accepted in revised form February 6, 1995

We give a functional Korovkin-type theorem on $B(X)$, the space of bounded complex-valued functions on an arbitrary set X and investigate a BKW-operator on $B(X)$ for a finite collection of test functions with a suitable property and a semi-norm defined by a finite subset of X.

1. INTRODUCTION AND RESULTS

The first author [6] and G. Anastassiou [2,3] independently proved the following discrete Korovkin theorem.

THEOREM A. *Let Y be a countable set, $B(Y)$ the space of real-valued bounded functions on Y with the usual supremum norm $\| \ \|_\infty$, $y_0 \in Y$ and $\{g_1, ..., g_k\}$ a finite subset of $B(Y)$ which has the property that there are real constants $\beta_1, ..., \beta_k$ such that $\sum_{i=1}^k \beta_i(g_i(y) - g_i(y_0)) \geqslant 1$ for all $y \neq y_0$. If $\{T_n\}$ is a sequence of positive linear operators on $B(Y)$ such that $T_n(\mathbf{1}) = \mathbf{1}$ for all $n \geqslant 1$ and $\lim_{n\to\infty}(T_n g_i)(y_0) = g_i(y_0)$ for $i = 1, ..., k$, then $\lim_{n\to\infty}(T_n f)(y_0) = f(y_0)$ for all $f \in B(Y)$, where $\mathbf{1}$ is the identity of $B(Y)$.*

We first give a simple proof of the above theorem by considering the Stone-Cech compactification of Y endowed with the discrete topology.

Throughout all sections except for the last section, let X be a set and $B(X)$ the Banach space of bounded complex-valued functions on X with the supremum norm. Let $E = \{x_1, ..., x_m\} \subset X$, $f_0 = \mathbf{1}$, the identity of $B(X)$, and let $\{f_1, ..., f_k\}$ be a finite collection of functions in $B(X)$ which satisfies the following two conditions:

0021-9045/96 $18.00

(i) There are complex constants $\beta_1, ..., \beta_k$ such that

$$\operatorname{Re} \sum_{i=1}^{k} \beta_i \left\{ f_i(x) - \frac{1}{m} \sum_{\gamma=1}^{m} f_i(x_\gamma) \right\} \begin{cases} \geqslant 1, & \text{if} \quad x \in X \setminus E \\ \geqslant 0, & \text{if} \quad x \in E; \end{cases}$$

and

$$\text{(ii)} \qquad \operatorname{rank} \begin{pmatrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_m) \\ \vdots & \cdots & \vdots \\ f_k(x_1) & \cdots & f_k(x_m) \end{pmatrix} = m.$$

Then we next prove the following Korovkin-type approximation theorem. This is a generalization of Theorem A and should be compared with Theorem 2.2.2 in the book by *F.* Altomare and M. Campiti [1], which is the analogon for positive approximation.

THEOREM 1. *Let* (μ, η) *be a pair of continuous linear functionals on* $B(X)$ *such that* $\eta(\mathbf{1}) = \|\mu\|$, $\eta(f) = \sum_{\gamma=1}^{m} \alpha_\gamma f(x_\gamma)$ $(\forall f \in B(X))$, *where* $\alpha_\gamma \in \mathbf{R}$ $(\gamma = 1, ..., m)$. *If* $\{\mu_\lambda\}$ *is a net of linear functionals on* $B(X)$ *such that* $\sup_\lambda \|\mu_\lambda\| \leqslant \|\mu\|$ *and* $\lim_\lambda \mu_\lambda(f_i) = \eta(f_i)$ *for* $i = 0, 1, ..., k$, *then* $\lim_\lambda \mu_\lambda(f) = \eta(f)$ *for all* $f \in B(X)$.

Remark. In view of the above theorem, the following question occurs: If μ is a continuous linear functional on $B(X)$ and if $\{T_\lambda\}$ is a net of linear contractions on $B(X)$ such that $\lim_\lambda \mu(T_\lambda f_i)$ exists for $i = 0, 1, ..., k$, then does $\lim_\lambda \mu(T_\lambda f)$ exist for all $f \in B(X)$? As observed in 4.2 of the last section, we negatively answer this question even if μ is an evaluation at a point in X.

Following [7, 8], we recall the definition of BKW-operators. A bounded linear operator T on $B(X)$ is said to be *BKW for test functions* $\{f_0, f_1, ..., f_k\}$ *and the seminorm* $\| \ \|_E$ if $\{T_\lambda\}$ is a net of bounded linear operators on $B(X)$ such that $\lim_\lambda \|T_\lambda\| = \|T\|$ and $\lim_\lambda \|T_\lambda(f_i) - T(f_i)\|_E = 0$ for $i = 0, 1, ..., k$, then $\lim_\lambda \|T_\lambda(f) - T(f)\|_E = 0$ for all $f \in B(X)$, where $\|f\|_E = \sup_{x \in E} |f(x)|$ $(f \in B(X))$. We note that the condition "$\lim_\lambda \|T_\lambda\| = \|T\|$" can be replaced by the condition "$\sup_\lambda \|T_\lambda\| \leqslant \|T\|$" in the above definition (cf. [8, Lemma 1.1]). A linear operator T on $B(X)$ will be called *locally unital* (*on* E) if $(T\mathbf{1})(x) = 1$ for every $x \in E$, and a *contraction* if $\|T\| \leqslant 1$. In particular, T is simply called *unital* if $(T\mathbf{1})(x) = 1$ for all $x \in X$. We will notice that

"T is a unital contraction" $\Leftrightarrow$ "T is positive and unital"

for any linear operator T on $B(X)$. The above theorem implies the following result which gives an information on locally unital linear contractions on $B(X)$ that are BKW for $\{f_0, f_1, ..., f_k\}$ and $\| \ \|_E$.

THEOREM 2. *Let T be a locally unital linear contraction on $B(X)$ such that*

$$\text{(I)} \qquad (Tf)(x) = \sum_{\gamma=1}^{m} f(x_\gamma)\, g_\gamma(x) \qquad (\forall x \in E, \forall f \in B(X))$$

for some $g_1, ..., g_m \in B(X)$. Then T is BKW for $\{f_0, f_1, ..., f_k\}$ and $\| \ \|_E$.

The preceding theorem implies that the average of certain homomorphisms from $B(X)$ into itself is BKW for $\{f_0, f_1, ..., f_k\}$ and $\| \ \|_E$ as observed in the following result.

COROLLARY 3. *Let $\{\phi_1, ..., \phi_N\}$ be a finite set consisting of maps from X into itself such that $\phi_i(E) \subset E$ $(i = 1, ..., N)$. For each i, let T_i be the composition operator on $B(X)$ defined by ϕ_i. Then the operator $(T_1 + \cdots + T_N)/N$ is a unital BKW-contraction on $B(X)$ for $\{f_0, f_1, ..., f_k\}$ and $\| \ \|_E$.*

Let βX be the Stone-Cech compactification of X endowed with the discrete topology so that we can regard $B(X)$ as the Banach space $C(\beta X)$ of all continuous functions on βX. Let T be a locally unital linear contraction on $B(X)$ and T^* its adjoint operator. Then for each $x \in E$, $T^*(\delta_x)$ is a probability Radon measure on βX, where δ_x denotes the Dirac measure concentrated at x. Note that the condition (I) in Theorem 2 is equivalent to the following condition:

(I′) $\operatorname{supp}(T^*\delta_x) \subset E$ for every $x \in E$.

Here "supp" denotes the support of a measure on βX. We further consider the conditions:

(II) There exists a point x_T in E such that $(T^*\delta_{x_T})(\beta X \backslash X) > 0$.

(III) There exists a point x_T in E and a point y in $X \backslash E$ such that $(T^*\delta_{x_T})(\{y\}) > 0$ and $\inf_{y \neq x \in X} \max_{1 \leqslant i \leqslant k} |f_i(x) - f_i(y)| = 0$.

Note that if T satisfies the condition (I) then, since $(T^*\delta_x)(\beta X \backslash E) = 0$ for every $x \in E$ by (I′), T satisfies neither of the conditions (II) and (III). The following result asserts that any locally unital linear contraction on $B(X)$ which satisfies (II) or (III) is not BKW for $\{f_0, f_1, ..., f_k\}$ and $\| \ \|_E$.

THEOREM 4. *Let T be a locally unital linear contraction on $B(X)$ which satisfies (II) or (III). Then there exists a sequence $\{T_n\}$ of unital linear contractions on $B(X)$ such that $\lim_{n \to \infty} \|T_n(f_i) - T(f_i)\|_E = 0$ for every $i = 0, 1, ..., k$ but $\lim_{n \to \infty}(T_n f)(x_T)$ fails to exist for some $f \in B(X)$.*

Remark. In case of $X = \mathbf{N}$, the natural numbers, $B(X)$ is the space l^∞ of all bounded sequence of complex numbers, and the unilateral backward shift operator T on l^∞ satisfies the condition (III), whenever each f_i is constant on $\mathbf{N}\backslash E$. On the other hand, if T is a linear operator on l^∞ such that

$$(T\{a_n\})_n = b_n(\operatorname*{Lim}_{n\to\infty} a_n) \qquad (n \in \mathbf{N},\ \{a_n\} \in l^\infty),$$

where $\mathrm{Lim}_{n\to\infty}$ denotes a Banach limit and $\{b_n\}$ is an element of l^∞ with $b_n = 1$ ($\forall n \in E$) and $|b_n| \leqslant 1$ ($\forall n \in \mathbf{N}$), then it is a locally unital contraction and satisfies the condition (II).

The following result asserts that all locally unital linear contractions on $B(X)$ that are BKW for $\{f_0, f_1, ..., f_k\}$ and $\|\ \|_E$ have the form (I) on E when no f_i is "notched".

THEOREM 5. *Assume that $f_1, ..., f_k$ satisfy the following condition*:

$$\text{(iii)} \quad \forall y \in X\backslash E, \exists\{y_1, y_2, ...\} \subset X \left| \begin{array}{ll} y_n \neq y & (n = 1, 2, ...), \\ \lim_{n\to\infty} f_i(y_n) = f_i(y) & (1 \leqslant i \leqslant k). \end{array} \right.$$

Then a locally unital linear contraction T on $B(X)$ is BKW for $\{f_0, f_1, ..., f_k\}$ and $\|\ \|_E$ if and only if it has the form (I) on E.

2. A SIMPLE PROOF OF THEOREM A

Let $\{T_n\}$ be a sequence of positive linear operators on $B(Y) \cong C(\beta Y)$ such that $T_n(\mathbf{1}) = \mathbf{1}$ for all $n \geqslant 1$ and $y_0 \in Y$. For each n, we consider a probability Radon measure μ_n on βY defined by $\mu_n(f) = (T_n f)(y_0)$ for every $f \in C(\beta Y)$. Since

$$\sum_{i=1}^{k} \beta_i\{g_i(y) - g_i(y_0)\} \geqslant 1 \qquad (y \neq y_0)$$

for the real constants $\beta_1, ..., \beta_k$ by hypothesis, the function $h = \sum_{i=1}^k \beta_i\{g_i - g_i(y_0)\mathbf{1}\}$ satisfies $h(\omega) \geqslant 1$ for every $\omega \in \beta Y$ with $\omega \neq y_0$. Suppose that $\lim_{n\to\infty} \mu_n(g_i) = g_i(y_0)$ for $i = 1, ..., k$. It follows that $\lim_{n\to\infty} \mu_n(h) = 0$. For $f \in B(Y)$ we have $|f(\omega) - f(y_0)| \leqslant 2\,\|f\|_\infty h(\omega)$ for all $\omega \in \beta Y$. Then

$$|\mu_n(f) - f(y_0)| = |\mu_n(f - f(y_0)\,\mathbf{1})| \leqslant 2\,\|f\|_\infty\, \mu_n(h) \to 0 \qquad (\text{as } n \to \infty),$$

and this finishes the proof. Q.E.D

Remark. Comparing the above proof with Theorem 1 in Nishishiraho [5] may be interesting.

3. Proof of Results

We recall that $E=\{x_1, ..., x_m\} \subset X$ and $\{f_1, ..., f_k\} \subset B(X)$ satisfy the conditions (i) and (ii), by hypothesis. Throughout this section, let h be the function in $B(X)$ defined by

$$h(x)=\operatorname{Re} \sum_{i=1}^{k} \beta_i \left\{ f_i(x) - \frac{1}{m} \sum_{\gamma=1}^{m} f_i(x_\gamma) \right\}$$

for every $x \in X$. Note that $h \mid E=0$. In fact,

$$\begin{aligned} \sum_{\zeta=1}^{m} h(x_\zeta) &= \operatorname{Re} \sum_{i=1}^{k} \sum_{\zeta=1}^{m} \beta_i \left\{ f_i(x_\zeta) - \frac{1}{m} \sum_{\gamma=1}^{m} f_i(x_\gamma) \right\} \\ &= \operatorname{Re} \sum_{i=1}^{k} \beta_i \left\{ \sum_{\zeta=1}^{m} f_i(x_\zeta) - \frac{m}{m} \sum_{\gamma=1}^{m} f_i(x_\gamma) \right\} \\ &= 0, \end{aligned}$$

so that $h(x_\zeta)=0$ for $\zeta=1, ..., m$, since $h \mid E \geqslant 0$ by the condition (i).

3.1. *Proof of Theorem* 1. Let (μ, η) be a pair of continuous linear functionals on $B(X)$ such that $\eta(\mathbf{1})=\|\mu\|$, $\eta(f)=\sum_{\gamma=1}^{m} \alpha_\gamma f(x_\gamma)$ $(\forall f \in B(X))$, where $\alpha_\gamma \in \mathbf{R}$ $(\gamma=1, ..., m)$. Suppose $\{\mu_\lambda\}$ is a net of linear functionals on $B(X)$ such that $\sup_\lambda \|\mu_\lambda\| \leqslant \|\mu\|$ and $\lim_\lambda \mu_\lambda(f_i)=\eta(f_i)$ for $i=0, 1, ..., k$ and let $f \in B(X)$ be any function. Then we have to show that $\lim_\lambda \mu_\lambda(f)=\eta(f)$. To do this let $\{\mu_{\lambda'}(f)\}$ be any subnet of $\{\mu_\lambda(f)\}$. Since $\|\mu_{\lambda'}\| \leqslant \|\mu\|$ for all λ', there exists a weak*-convergent subnet $\{\mu_{\lambda''}\}$ of $\{\mu_{\lambda'}\}$. Let $\tilde{\mu}$ be the weak*-limit of $\{\mu_{\lambda''}\}$, so that $\|\tilde{\mu}\| \leqslant \|\mu\|$. Also since $\tilde{\mu}(\mathbf{1})=\lim_{\lambda''} \mu_{\lambda''}(\mathbf{1})=\eta(\mathbf{1})=\|\mu\|$, it follows that $\tilde{\mu}$ is positive. Note that

$$\tilde{\mu}(f_i)=\lim_{\lambda''} \mu_{\lambda''}(f_i)=\eta(f_i)$$

for $i=0, 1, ..., k$. Then we have

$$\begin{aligned} \tilde{\mu}(h) &= \operatorname{Re} \tilde{\mu}\left(\sum_{i=1}^{k} \beta_i \left\{ f_i - \frac{1}{m} \sum_{\gamma=1}^{m} f_i(x_\gamma) \right\} \right) \\ &= \operatorname{Re} \eta \left(\sum_{i=1}^{k} \beta_i \left\{ f_i - \frac{1}{m} \sum_{\gamma=1}^{m} f_i(x_\gamma) \right\} \right) \\ &= \eta(h) = \sum_{\zeta=1}^{m} \alpha_\zeta h(x_\zeta) \\ &= 0 \qquad (\text{since } h=0 \text{ on } E=\{x_1, ..., x_m\}). \end{aligned}$$

Now by the condition (ii), we can find complex constants $c_0, c_1, ..., c_k$ such that

$$f(x) = \sum_{i=0}^{k} c_i f_i(x)$$

for every $x \in E$. Set $g = f - \sum_{i=0}^{k} c_i f_i$. Since $g \mid E = 0$ and $h(x) \geqslant 1$ for every $x \in X \setminus E$ by condition (i), it follows that

$$|\tilde{\mu}(g)| \leqslant \tilde{\mu}(|g|) \leqslant \tilde{\mu}(\|g\| \, h) = \|g\| \, \tilde{\mu}(h) = 0;$$

hence we have

$$\begin{aligned} \lim_{\lambda''} \mu_{\lambda''}(f) = \tilde{\mu}(f) &= \sum_{i=0}^{k} c_i \tilde{\mu}(f_i) = \sum_{i=0}^{k} c_i \eta(f_i) \\ &= \eta\left(\sum_{i=0}^{k} c_i f_i\right) = \eta\left(g + \sum_{i=0}^{k} c_i f_i\right) = \eta(f). \end{aligned}$$

In other words, $\lim_{\lambda} \mu_{\lambda}(f) = \eta(f)$. Q.E.D

3.2. *Proof of Theorem* 2. Let T be a locally unital linear contraction on $B(X)$ which has the form (I) on E, i.e.,

$$(Tf)(x) = \sum_{\gamma=1}^{m} f(x_\gamma) \, g_\gamma(x) \qquad (\forall x \in E, \forall f \in B(X))$$

for some $g_1, ..., g_m \in B(X)$. Here we note that if $x \in E$ then $g_\gamma(x) = (T^* \delta_x)(\{x_\gamma\}) \geqslant 0$ for every $x_\gamma \in E = \{x_1, ..., x_m\}$ and $\sum_{\gamma=1}^{m} g_\gamma(x) = 1$, because $T^*(\delta_x)$ is probability Radon measure on βX which follows from the fact that $(T\mathbf{1})(x) = 1$. Suppose $\{T_\lambda\}$ is a net of linear contractions on $B(X)$ such that $\lim_\lambda \|T_\lambda(f_i) - T(f_i)\|_E = 0$ for $i = 0, 1, ..., k$. Let $f \in B(X)$ and $x_\zeta \in E$ be fixed arbitrarily. Consider the functional η on $B(X)$ defined by

$$\eta(g) = (Tg)(x_\zeta) \left(= \sum_{\gamma=1}^{m} g(x_\gamma) \, g_\gamma(x_\zeta) \right)$$

for every $g \in B(X)$. Then we have $\eta(\mathbf{1}) = \sum_{\gamma=1}^{m} g_\gamma(x_\zeta) = (T\mathbf{1})(x_\zeta) = 1$. Let μ be the evaluation at x_ζ and so $\eta(\mathbf{1}) = 1 = \|\mu\|$. Moreover,

$$\lim_{\lambda} \mu(T_\lambda f_i) = \lim_{\lambda} (T_\lambda f_i)(x_\zeta) = (Tf_i)(x_\zeta) = \eta(f_i)$$

for $i = 0, 1, ..., k$. Therefore, by Theorem 1, we have

$$\lim_{\lambda} (T_\lambda f)(x_\zeta) = \eta(f) = \sum_{\gamma=1}^{m} f(x_\gamma) \, g_\gamma(x_\zeta) = (Tf)(x_\zeta).$$

Since E is a finite set, it follows that $\lim_\lambda \|T_\lambda(f)-T(f)\|_E=0$. In other words, T is BKW for $\{f_0, f_1, ..., f_k\}$ and $\|\ \|_E$. Q.E.D

3.3. *Proof of Corollary* 3. Let $\{\phi_1, ..., \phi_N\}$ be a finite set consisting of maps from X into itself such that $\phi_i(E)\subset E$ $(i=1, ..., N)$. For each i, let T_i be the composition operator on $B(X)$ defined by ϕ_i and set $T=(T_1+\cdots+T_N)/N$. Then it is obvious that T is a unital linear contraction on $B(X)$. For each $x\in X$ and $1\leqslant\gamma\leqslant m$, let

$$g_\gamma(x)=\frac{\#\{1\leqslant i\leqslant N : \varphi_i(x)=x_\gamma\}}{N}.$$

Then each g_γ is a function in $B(X)$ and we can easily see that

$$(Tf)(x)=\sum_{\gamma=1}^{m} f(x_\gamma)\, g_\gamma(x)$$

for every $x\in E$ and $f\in B(X)$. Hence the corollary follows from Theorem 2. Q.E.D

3.4. *Proof of Theorem* 4. Case (a). Let T be a locally unital linear contraction on $B(X)$ satisfying the condition (II). Since βX is totally disconnected, it is zero dimensional (i.e., the clopen sets form a base for βX) (cf. [4, Theorem 3.5]). Then for each $n\geqslant 1$, we can find a finite collection $\Delta_n=\{Y^n_1, ..., Y^n_{\alpha(n)}\}$ of pairwise disjoint non-empty clopen sets in βX such that

$$\beta X=Y^n_1\cup\cdots\cup Y^n_{\alpha(n)} \qquad \text{and} \qquad |f_i(x)-f_i(y)|<\frac{1}{n} \qquad (0\leqslant i\leqslant k)$$

for all $x, y\in Y^n_j$, $j=1, ..., \alpha(n)$. Here, if necessary, taking a common refinement of Δ_n and Δ_{n+1}, we may assume without loss of generality that Δ_{n+1} is a refiment of Δ_n for each n. Note that each $Y^n_j\cap X$ is a non-empty set, hence choose a point y^n_j in $Y^n_j\cap X$ and set $B_n=\{y^n_1, ..., y^n_{\alpha(n)}\}$. Since each Y^n_j is clopen in βX and X is dense in βX, it follows that if $Y^n_j\cap(\beta X\backslash X)$ is a non-empty set, then $Y^n_j\cap X$ is an infinite set. Therefore we can choose again these points y^n_j so that if $m<n$ and $Y^n_j\cap(\beta X\backslash X)\neq\varnothing$, then $y^n_j\notin B_m$. We consider the sequence $\{T_n\}$ of linear operators on $B(X)$ defined by

$$(T_nf)(x)=\begin{cases} f(x), & \text{if } \ x\in X\backslash E \\ \displaystyle\sum_{j=1}^{\alpha(n)} f(y^n_j)(T^*\delta_x)(Y^n_j), & \text{if } \ x\in E \end{cases}$$

for every $f \in B(X)$ and $n \geqslant 1$. Since T is locally unital (on E) by hypothesis, it follows that each T_n is unital and positive. Moreover,

$$|(T_n f_i)(x) - (Tf_i)(x)| \leqslant \sum_{j=1}^{\alpha(n)} \int_{Y_j^n} |f_i(y_j^n) - f_i(\omega)| \, d(T^*\delta_x)(\omega)$$

$$\leqslant \sum_{j=1}^{\alpha(n)} \frac{1}{n} (T^*\delta_x)(Y_j^n) = \frac{1}{n}$$

for each $n \geqslant 1$, $x \in E$ and $0 \leqslant i \leqslant k$. After taking the limit with respect to n, we see that $\lim_{n \to \infty} (T_n f_i)(x) = (Tf_i)(x)$ for each $x \in E$ and $0 \leqslant i \leqslant k$, and hence the finiteness of E implies that

$$\lim_{n \to \infty} \|T_n(f_i) - T(f_i)\|_E = 0$$

for $i = 0, 1, ..., k$. Now let for each $n \geqslant 1$

$$W_n = \bigcup \{ Y_j^n : Y_j^n \cap (\beta X \backslash X) \neq \varnothing \}$$

and

$$A_n = \{ y_j^n \in B_n : Y_j^n \cap (\beta X \backslash X) \neq \varnothing \}.$$

Since Δ_{n+1} is a refinement of Δ_n for each $n \geqslant 1$, it follows that

$$W_1 \supset W_2 \supset \cdots \supset \beta X \backslash X,$$

and by setting $W = \bigcap_{n=1}^{\infty} W_n$ we have

$$(T^*\delta_{x_T})(W) = \lim_{n \to \infty} (T^*\delta_{x_T})(W_n) \geqslant (T^*\delta_{x_T})(\beta X \backslash X) > 0$$

from the condition (II). Thus we can choose an integer N so that

$$(T^*\delta_{x_T})(W_n) < \tfrac{4}{3} (T^*\delta_{x_T})(W)$$

for all $n \geqslant N$. Next, note that $A_n \neq \varnothing$ $(n = 1, 2, \ldots)$ and $A_n \cap B_m = \varnothing$ when $m < n$. Thus $A_1, A_2, \ldots$ are pairwise disjoint non-empty sets in X and hence we can consider the function f in $B(X)$ defined by

$$f(x) = \begin{cases} (-1)^n, & \text{if} \quad x \in A_n \quad \text{for some } n \geqslant N \\ 0, & \text{if} \quad x \in X \Big\backslash \bigcup_{n=N}^{\infty} A_n. \end{cases}$$

Recall that $x_T \in E$, and thus by the definition of T_n we get for all $n \geqslant N$

$$(T_n f)(x_T) = \sum_{y_j^n \in A_n} (-1)^n (T^*\delta_{x_T})(Y_j^n) + \sum_{y_j^n \in A_N \cup \cdots \cup A_{n-1}} f(y_j^n)(T^*\delta_{x_T})(Y_j^n).$$

But since

$$\sum_{y_j^n \in A_n} (T^*\delta_{x_T})(Y_j^n) = (T^*\delta_{x_T})(W_n) \geqslant (T^*\delta_{x_T})(W)$$

and

$$\sum_{y_j^n \in A_N \cup \cdots \cup A_{n-1}} (T^*\delta_{x_T})(Y_j^n) \leqslant (T^*\delta_{x_T})(W_N \backslash W_n) < \tfrac{1}{3}(T^*\delta_{x_T})(W),$$

it follows that

$$(T_n f)(x) \begin{cases} \geqslant \frac{2}{3}(T^*\delta_{x_T})(W), & \text{if } n \text{ is even} \\ \leqslant -\frac{2}{3}(T^*\delta_{x_T})(W), & \text{if } n \text{ is odd,} \end{cases}$$

which proves that $\lim_{n\to\infty}(T_n f)(x_T)$ does not exist because $(T^*\delta_{x_T})(W) > 0$.

Case (b). Let T be a locally unital linear contraction on $B(X)$ satisfying the condition (III). Then for each $n \geqslant 1$, we can choose a point y_n in X such that $y_n \neq y$ and $\max_{1 \leqslant i \leqslant k} |f_i(y_n) - f_i(y)| < 1/n$. Suppose first that $\{y_1, y_2, ..., \}$ is an infinite set. We may assume that $y_1, y_2, \ldots$ are mutually different. For each $n \geqslant 1$ and $f \in B(X)$, we set

$$(T_n f)(x) = \begin{cases} f(x), & \text{if} \quad x \in X \backslash E \\ \int_{\beta X \backslash \{y\}} f \, d(T^*\delta_x) + f(y_n)(T^*\delta_x)(\{y\}), & \text{if} \quad x \in E, \end{cases}$$

so that $\{T_n\}$ is a sequence of unital linear contractions on $B(X)$ such that

$$\lim_{n\to\infty} \|T_n(f_i) - T(f_i)\|_E = 0$$

for $i = 0, 1, ..., k$. However, for any function f in $B(X)$ such that $f(y_n) = (-1)^n$ for each $n \geqslant 1$, $\lim_{n\to\infty}(T_n f)(x_T)$ does not exist, since

$$(T_n f)(x_T) = \int_{\beta X \backslash \{y\}} f \, d(T^*\delta_{x_T}) + (-1)^n (T^*\delta_{x_T})(\{y\})$$

for each $n \geqslant 1$ and $(T^*\delta_{x_T})(\{y\}) > 0$ by the condition (III).

Suppose next that $\{y_1, y_2, ...\}$ is a finite set. Then we can find a point z in X such that $z \neq y$ and $f_i(z) = f_i(y)$ for $i = 0, 1, ..., k$. For each $n \geqslant 1$ and $f \in B(X)$, we set

$$(T_n f)(x) = \begin{cases} f(x), & \text{if } x \in X \setminus E \\ (Tf)(x), & \text{if } x \in E \text{ and } n \text{ is even} \\ \int_{\beta X \setminus \{y\}} f \, d(T^*\delta_x) + f(z)(T^*\delta_x)(\{y\}), & \text{if } x \in E \text{ and } n \text{ is odd,} \end{cases}$$

so that $\{T_n\}$ is a sequence of unital linear contractions on $B(X)$ such that $(T_n f_i)(x) = (Tf_i)(x)$ for every $x \in E$ and $0 \leqslant i \leqslant k$. However, for any function f in $B(X)$ such that $f(z) \neq f(y)$, $\lim_{n \to \infty}(T_n f)(x_T)$ does not exist, since

$$|(T_{2n-1} f)(x_T) - (T_{2n} f)(x_T)| = |f(z) - f(y)| \, (T^*\delta_{x_T})(\{y\})$$

for each $n \geqslant 1$ and $(T^*\delta_{x_T})(\{y\}) > 0$ by the condition (III). Q.E.D

3.5. *Proof of Theorem* 5. Assume that $f_1, ..., f_k$ satisfy the condition (iii) and let T be a locally unital linear contraction on $B(X)$. If T has the form (I) on E, then it is BKW for $\{f_0, f_1, ..., f_k\}$ and $\| \ \|_E$ from Theorem 2. To show the converse suppose that T does not have the form (I) on E and hence there exists a point z in E such that $\mathrm{supp}(T^*\delta_z) \not\subset E$. If $(T^*\delta_z)(\beta X \setminus X) > 0$, then T is not BKW for $\{f_0, f_1, ..., f_k\}$ and $\| \ \|_E$ from Theorem 4. If $(T^*\delta_z)(\beta X \setminus X) = 0$, then there exists a point y in $X \setminus E$ such that $(T^*\delta_z)(\{y\}) > 0$ because $T^*(\delta_z)$ is a regular measure with $\mathrm{supp}(T^*\delta_z) \not\subset E$. Then we have $\inf_{y \neq x \in X} \max_{1 \leqslant i \leqslant k} |f_i(x) - f_i(y)| = 0$ by the condition (iii) and hence T is not BKW for $\{f_0, f_1, ..., f_k\}$ and $\| \ \|_E$ from Theorem 4. Q.E.D

4. Examples

4.1. *An example of* $\{f_1, ..., f_k\}$ *satisfying conditions* (i) *and* (ii). Let $X = \mathbf{C}$, the complex numbers, and $E = \{z_1, ..., z_m\}$, where $m \geqslant 2$ and $z_i \neq z_j$ $(i \neq j)$. For any finite sequence $\{\alpha_1, ..., \alpha_{m-1}\}$ of complex numbers, define

$$f_i(z) = \begin{cases} z^i, & \text{if } z \in E \\ \alpha_i, & \text{if } z \in X \setminus E \end{cases} \qquad (1 \leqslant i \leqslant m-1). \tag{1}$$

We show that there is a finite sequence $\{\alpha_1, ..., \alpha_{m-1}\}$ such that the corresponding functions $f_1, ..., f_{m-1}$ in $B(X)$ satisfy the conditions (i) and (ii) in the first section. We note, as is well-known, that without any additional hypothesis the functions $f_1, ..., f_{m-1}$ always satisfy the condition (ii). To find a sequence $\{\alpha_1, ..., \alpha_{m-1}\}$ such that the corresponding functions $f_1, ..., f_{m-1}$ satisfy the condition (i), let

$$a_i = \frac{1}{m} \sum_{\gamma=1}^{m} f_i(z_\gamma) \qquad (1 \leqslant i \leqslant m-1) \tag{2}$$

and

$$A = \begin{pmatrix} f_1(z_1) - a_1 & \cdots & f_1(z_{m-1}) - a_1 \\ \vdots & \vdots & \vdots \\ f_{m-1}(z_1) - a_{m-1} & \cdots & f_{m-1}(z_{m-1}) - a_{m-1} \end{pmatrix}. \tag{3}$$

By an elementary calculation we observe that rank $A = m-1$. Thus there exists a unique solution $(d_1, ..., d_{m-1})$ in $\mathbf{C}^{m-1}$ for the equation

$$\begin{pmatrix} f_1(z_m) - a_1 \\ \vdots \\ f_{m-1}(z_m) - a_{m-1} \end{pmatrix} = A \begin{pmatrix} d_1 \\ \vdots \\ d_{m-1} \end{pmatrix}. \tag{4}$$

Using this solution $(d_1, ..., d_{m-1})$, we first prove the following

LEMMA. *There exists a vector* $(\beta_1, ..., \beta_{m-1})$ *in* $\mathbf{C}^{m-1}$, *with* $(\beta_1, ..., \beta_{m-1}) \neq (0, ..., 0)$, *such that the function*

$$h(z) = \operatorname{Re} \sum_{i=1}^{m-1} \beta_i (f_i(z) - a_i) \qquad (z \in X) \tag{5}$$

satisfies $h \mid E = 0$.

Proof. Case (a). Suppose there is a number $1 \leqslant i \leqslant m-1$ such that $d_i \in \mathbf{R}$, the real numbers. Then define a vector $(\beta_1, ..., \beta_{m-1})$ in $\mathbf{C}^{m-1}$ by the following relation

$$(\beta_1, ..., \beta_{m-1})\, A = \sqrt{-1}\, \mathbf{e}_i, \tag{6}$$

where $\mathbf{e}_i$ is the (row) vector in $\mathbf{C}^{m-1}$ whose i-th coordinate is 1 and whose other coordinate are all 0. It follows from (3), (4) and (5) that $h \mid E = 0$.

Case (b). Suppose $m \geqslant 3$ and $d_i \notin \mathbf{R}$ for all $1 \leqslant i \leqslant m-1$. Then in particular, $d_1, d_2 \notin \mathbf{R}$ and we can choose two complex numbers c_1 and c_2 so that

$$(c_1, c_2) \neq (0, 0) \qquad \text{and} \qquad \operatorname{Re}(c_1) = \operatorname{Re}(c_2) = \operatorname{Re}(c_1 d_1 + c_2 d_2) = 0. \tag{7}$$

Define a vector $(\beta_1, ..., \beta_{m-1})$ in $\mathbf{C}^{m-1}$ by the relation

$$(\beta_1, ..., \beta_{m-1})\, A = (c_1, c_2, 0, ..., 0). \tag{8}$$

It follows that $(\beta_1, ..., \beta_{m-1}) \neq (0, ..., 0)$ and $h \mid E = 0$.

Case (c). Suppose $m = 2$. It follows from (1) and (2) that $f_1(z_1) - a_1 \neq 0$ and $\sum_{i=1}^{2} (f_1(z_i) - a_1) = 0$. Hence $d_1 = -1$ is a unique solution of the equation (4), and so this is a part of Case (a). Q.E.D

By the lemma, we write

$$\alpha_i = \bar{\beta}_i + \frac{1}{m} \sum_{\gamma=1}^{m} f_i(z_\gamma) \qquad (1 \leqslant i \leqslant m-1), \tag{9}$$

where $\overline{\beta}_i$ denotes the complex conjugate of β_i (and we may assume without loss of generality that $\sum_{i=1}^{m-1} |\beta_i|^2 \geqslant 1$). Then the corresponding functions $f_1, ..., f_{m-1}$ for $\{\alpha_1, ..., \alpha_{m-1}\}$ satisfy the conditions (i) and (ii) as required. We can give a shorter and easier proof in case of $k \geqslant m$.

4.2. *An example of a sequence* $\{T_n\}$ *of unital contractions for which* $\lim_{n \to \infty}(T_n f_i)(x)$ *exists for all* $1 \leqslant i \leqslant k$ *but such that* $\lim_{n \to \infty}(T_n f)(x)$ *fails to exists for some* f *in* $B(X)$. Let X, $E = \{z_1, ..., z_m\}$ and $\{f_1, ..., f_{m-1}\}$ be as in 4.1 and define

$$(T_n f)(z) = f(z+n) \tag{10}$$

for each $n \geqslant 1$ and $f \in B(X)$. Then $\{T_n\}$ is a sequence of unital linear contractions on $B(X)$. Let μ be the evaluation at the origin of X. Then $\mu(T_n \mathbf{1}) = 1$ for all $n \geqslant 1$ and $\lim_{n \to \infty} \mu(T_n f_i) = \alpha_i$ for all $1 \leqslant i \leqslant k$. However, $\lim_{n \to \infty} \mu(T_n f)$ fails to exist for any function f in $B(X)$ such that $f(n) = (-1)^n$ for all natural numbers n.

4.3. *An example of* X *and* $E = \{x_1, ..., x_m\}$ *such that all locally unital linear contractions on* $B(X)$ *are* BKW *for* $\{\mathbf{1}, f_1, ..., f_k\}$ *and* $\|\ \|_E$ *whenever* $\{f_1, ..., f_k\}$ *satisfies conditions* (i) *and* (ii). Let $X = \{x_1, ..., x_{m+1}\}$, $E = \{x_1, ..., x_m\}$, $f_0 = \mathbf{1}$ and let $\{f_1, ..., f_k\}$ be a finite collection of functions in $B(X)$ which satisfies the conditions (i) and (ii) in the first section. Then an arbitrary locally unital linear contraction T on $B(X)$ is BKW for $\{\mathbf{1}, f_1, ..., f_m\}$ and $\|\ \|_E$. Actually let $\{T_\lambda\}$ be a net of contractions on $B(X)$ such that $\lim_\lambda \|T_\lambda(f_i) - T(f_i)\|_E = 0$ for $i = 0, 1, ..., m$. Let $y \in E$ and $f \in B(X)$ be fixed arbitrarily. Then we have only to show that $\lim_\lambda (T_\lambda f)(y) = (Tf)(y)$. As observed in the proof of Theorem 1, since $T(\mathbf{1}) = 1$ on E and $y \in E$, we can assume that $\{T_\lambda^*(\delta_y)\}$ converges in the weak * topology to a probability measure on βX, say η. Since

$$\eta(f_i) = \lim_\lambda (T_\lambda f_i)(y) = (Tf_i)(y) = (T^* \delta_y)(f_i) \qquad (0 \leqslant i \leqslant k),$$

it follows that η and $T^*(\delta_y)$ agree on the linear span of $\{f_0, f_1, ..., f_k\}$, hence $\eta(h) = (T^*\delta_y)(h) = (Th)(y)$ because η and $T^*(\delta_y)$ are positive. By the condition (ii), there are scalars $c_0, c_1, ..., c_k$ such that $f(x) = \sum_{i=0}^k c_i f_i(x)$ for every $x \in E$. Note that $h \mid E = 0$ and $h(x_{m+1}) \geqslant 1$, hence by setting

$$c = \frac{1}{h(x_{m+1})} \left\{ f(x_{m+1}) - \sum_{i=0}^{k} c_i f_i(x_{m+1}) \right\},$$

we have $f = ch + \sum_{i=0}^k c_i f_i$. Therefore

$$\begin{aligned}\lim_{\lambda} (T_\lambda f)(y) &= \eta(f) = c\eta(h) + \sum_{i=0}^{k} c_i \eta(f_i) \\ &= c(Th)(y) + \sum_{i=0}^{k} c_i (Tf_i)(y) = (Tf)(y)\end{aligned}$$

as required.

4.4. *An example of X, $E = \{x_1, ..., x_m\}$, and $\{f_1, ..., f_k\}$ for which the converses of Theorems* 2 *and* 4 *do not hold.* Throughout the remaider of this section, let X be a set containing $m+2$-points; $x_1, ..., x_m, x_{m+1}, x_{m+2}$, $E = \{x_1, ..., x_m\}$ and $\{f_1, ..., f_m\}$ a subset of $B(X)$ defined by

$$f_1(x) = \begin{cases} 1, & \text{if} \quad x = x_1 \\ 0, & \text{if} \quad x = x_\gamma \quad (2 \leqslant \gamma \leqslant m) \\ 3, & \text{if} \quad x = x_{m+1} \\ 2, & \text{otherwise} \end{cases}$$

and

$$f_i(x) = \begin{cases} 1, & \text{if} \quad x = x_i \\ 0, & \text{if} \quad x \neq x_i \end{cases} \qquad (i = 2, ..., m).$$

Let h be a function in $B(X)$ defined by

$$h = \left(\sum_{i=1}^{m} f_i \right) - \mathbf{1}, \tag{11}$$

so that

$$h \mid E = 0, \quad h(x_{m+1}) = 2 \quad \text{and} \quad h = 1 \text{ on } X \backslash (E \cup \{x_{m+1}\}). \tag{12}$$

Accordingly $\{f_1, ..., f_m\}$ satisfies the conditions (i) and (ii) in the first section. Moreover we note that

$$\inf_{x_{m+1} \neq x \in X} \max_{1 \leqslant i \leqslant m} |f_i(x) - f_i(x_{m+1})| = 1. \tag{13}$$

4.4.1. Consider the following unital linear contraction T on $B(X)$ defined by

$$(Tf)(x)=\begin{cases} f(x), & \text{if} \quad x\neq x_1 \\ \dfrac{f(x_1)+f(x_{m+1})}{2}, & \text{if} \quad x=x_1. \end{cases} \tag{14}$$

Then $(T^*\delta_{x_1})(\beta X\backslash X)=0$ and $(T^*\delta_{x_1})(\{x_{m+1}\})=\frac{1}{2}$, hence T satisfies neither of the conditions (II) and (III). Moreover we see that T is not BKW for $\{\mathbf{1}, f_1, ..., f_m\}$ and $\|\ \|_E$. In fact, set

$$(T_{2n}f)(x)=\begin{cases} f(x), & \text{if} \quad x\neq x_1 \\ \dfrac{f(x_1)+f(x_{m+1})+f(x_{m+2})}{3}, & \text{if} \quad x=x_1 \end{cases}$$

and

$$T_{2n-1}(f)=T(f)$$

for each $f\in B(X)$ and $n\geqslant 1$. Then $\{T_n\}$ is a sequence of unital linear contractions on $B(X)$ such that $\lim_{n\to\infty}(T_nf)(x)=(Tf)(x)$ for all $f\in B(X)$ and $x\in X\backslash\{x_1\}$. Further by (14) we see that

$$\lim_{n\to\infty}(T_nf_i)(x_1)=2\delta_{i,1}=(Tf_i)(x_1) \qquad (1\leqslant i\leqslant m),$$

where $\delta_{i,j}$ is Kronecker's delta function. But $\lim_{n\to\infty}(T_nf)(x_1)$ fails to exists for the following function f in $B(X)$ defined by

$$f(x)=\begin{cases} 1, & \text{if} \quad x=x_1 \\ 0, & \text{if} \quad x\neq x_1. \end{cases}$$

4.4.2. Let $m\geqslant 2$ and define

$$(Tf)(x)=\begin{cases} f(x), & \text{if} \quad x\neq x_1 \\ \dfrac{f(x_2)+f(x_{m+1})}{2}, & \text{if} \quad x=x_1. \end{cases} \tag{15}$$

Then T is a unital linear contraction on $B(X)$ with $\mathrm{supp}(T^*\delta_{x_1})=\{x_2, x_{m+1}\}$, and hence T does not satisfy the condition (I). However T is BKW for $\{\mathbf{1}, f_1, ..., f_m\}$ and $\|\ \|_E$. Actually let $\{T_\lambda\}$ be a net of contractions on $B(X)$ such that $\lim_\lambda \|T_\lambda(f_i)-T(f_i)\|_E=0$ for $i=0, 1, ..., m$, where $f_0=\mathbf{1}$. Then by Theorem 1, we have

$$\lim_\lambda (T_\lambda f)(x_\gamma)=(Tf)(x_\gamma)$$

for all $f \in B(X)$ and $2 \leqslant \gamma \leqslant m$. It only remains to show that

$$\lim_{\lambda} (T_{\lambda} f)(x_1) = (Tf)(x_1)$$

for all $f \in B(X)$. As in 4.3, we can assume that $\{T_{\lambda}^*(\delta_{x_1})\}$ converges in the weak* topology to a probability Radon measure on βX, say η. Then by (11), (12), and (15),

$$\begin{aligned} \eta(h) &= \lim_{\lambda} \sum_{i=1}^{m} (T_{\lambda} f_i)(x_1) - \lim_{\lambda} (T_{\lambda} \mathbf{1})(x_1) \\ &= \sum_{i=1}^{m} (Tf_i)(x_1) - (T\mathbf{1})(x_1) \qquad (16) \\ &= (Th)(x_1) = 1. \end{aligned}$$

Also, $\eta(f_2) = (Tf_2)(x_1) = \frac{1}{2}$ and $\eta(f_i) = (Tf_i)(x_1) = 0$ for each $3 \leqslant i \leqslant m$; hence $\eta(\{x_2\}) = \frac{1}{2}$ and $\eta(\{x_3, ..., x_m\}) = 0$. Further by (12) and (16),

$$1 = \int_{\beta X} h \, d\eta = 2\eta(\{x_{m+1}\}) + \eta(\beta X \setminus \{x_1, ..., x_m, x_{m+1}\}). \qquad (17)$$

On the other hand, since $x_2, x_{m+1} \notin \beta X \setminus \{x_1, ..., x_m, x_{m+1}\}$ and $x_2 \neq x_{m+1}$, we have

$$\eta(\{x_{m+1}\}) + \eta(\beta X \setminus \{x_1, ..., x_m, x_{m+1}\}) \leqslant 1 - \eta(\{x_2\}) = \tfrac{1}{2}. \qquad (18)$$

Combining (17) and (18), we obtain $\eta(\{x_{m+1}\}) = 1/2$ and so $\eta = (\delta_{x_2} + \delta_{x_{m+1}})/2$. In other words, $\lim_{\lambda} (T_{\lambda} f)(x_1) = (Tf)(x_1)$ for all $f \in B(X)$.

Acknowledgment

The authors thank the referee for simplifying the original proofs and for other helpful comments.

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